

Lagrangian Dynamics of Collisionless Matter

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Abstract. The non-linear dynamics of self-gravitating irrotational dust is analyzed in a general relativistic framework, using synchronous and comoving coordinates. Writing the equations in terms of the metric tensor of the spatial sections orthogonal to the fluid flow allows an unambiguous expansion in inverse powers of the speed of light. The Newtonian and post-Newtonian approximations are derived in Lagrangian form. A general formula for the gravitational waves generated by the non-linear evolution of cosmological perturbations is given. It is argued that a stochastic gravitational-wave background is produced by non-linear cosmic structures, with present-day closure density $\Omega_{gw} \sim 10^{-5} - 10^{-6}$ on Mpc scale.

1. Introduction

The gravitational instability of collisionless matter in a cosmological framework is usually studied within the Newtonian approximation, which basically consists in neglecting terms higher than the first in metric perturbations around a matter-dominated Friedmann–Robertson–Walker (FRW) background, while keeping non-linear density and velocity perturbations. This approximation is usually thought to produce accurate results in a wide spectrum of cosmological scales, namely on scales much larger than the Schwarzschild radius of collapsing bodies and much smaller than the Hubble horizon scale, where the peculiar gravitational potential φ_g , divided by the square of the speed of light c^2 to obtain a dimensionless quantity, keeps much less than unity, while the peculiar matter flow never becomes relativistic. To be more specific, the Newtonian approximation consists in perturbing only the time–time component of the FRW metric tensor by an amount $2\varphi_g/c^2$, where φ_g is related to the matter density fluctuation δ via the cosmological Poisson equation, $\nabla_x^2 \varphi_g(\vec{x}, \tau) = 4\pi G a^2(\tau) \varrho_b(\tau) \delta(\vec{x}, \tau)$, where ϱ_b is the background matter density, $a(\tau)$ the appropriate FRW scale-factor and τ the conformal time. The fluid dynamics is then usually studied in Eulerian coordinates by accounting for mass conservation and using the cosmological version of the Euler equation for a self-gravitating pressureless fluid to close the system. To motivate the use of this “hybrid approximation”, which deals with perturbations of the matter and the geometry at a different perturbative order, one can either formally expand the correct equations of General Relativity (GR) in inverse powers of the speed of light or simply notice that the peculiar gravitational potential is strongly suppressed with respect to the matter perturbation by the square of the ratio of the perturbation scale λ to the Hubble radius $r_H = cH^{-1}$ (H being the Hubble constant): $\varphi_g/c^2 \sim \delta (\lambda/r_H)^2$.

Such a simplified approach, however, already fails in producing an accurate description of the trajectories of relativistic particles, such as photons. Neglecting the relativistic perturbation of the space-space components of the metric, which in the so-called longitudinal gauge is just $-2\varphi_g/c^2$, would imply a mistake by a factor of two in well-known effects such as the Sachs-Wolfe, Rees-Sciama and gravitational lensing. The level of accuracy not only depends on the peculiar velocity of the matter producing the spacetime curvature, but also on the nature of the particles carrying the signal to the observer. Said this way, it may appear that the only relativistic correction required to the usual Eulerian Newtonian picture is that of writing the metric tensor in the “weak field” form (e.g. Peebles 1993)

$$ds^2 = a^2(\tau) \left[- \left(1 + \frac{2\varphi_g}{c^2} \right) c^2 d\tau^2 + \left(1 - \frac{2\varphi_g}{c^2} \right) dl^2 \right]. \quad (1)$$

As we are going to show, this is not the whole story. It is well-known in fact that the gravitational instability of aspherical perturbations (which is the generic case) leads to the formation of very anisotropic structures whenever pressure gradients can be neglected (e.g. Shandarin et al. 1995 and references therein). Matter first flows in almost two-dimensional structures called pancakes, which then merge and fragment to eventually form one-dimensional filaments and point-like clumps. During the process of pancake formation the matter density, the shear and the tidal field formally become infinite along evanescent two-dimensional configurations corresponding to caustics; after this event a number of highly non-linear phenomena, such as vorticity generation by multi-streaming, merging, tidal disruption and fragmentation, occur. Most of the pathology of the caustic formation process, such as the local divergence of the density, shear and tide, and the formation of multi-stream regions, are just an artifact of extrapolating the pressureless fluid approximation beyond the point at which pressure gradients and viscosity become important. In spite of these limitations, however, it is generally believed that the general anisotropy of the collapse configurations, either pancakes or filaments, is a generic feature of cosmological structures originated through gravitational instability, which would survive even in the presence of a collisional component.

This simple observation shows the inadequacy of the standard Newtonian paradigm. According to it the lowest scale at which the approximation can be reasonably applied is set by the amplitude of the gravitational potential and is given by the Schwarzschild radius of the collapsing body, which is negligibly small for any relevant cosmological mass scale. What is completely missing in this criterion is the role of the shear which causes the presence of non-scalar contributions to the metric perturbations. A non-vanishing shear component is in fact an unavoidable feature of realistic cosmological perturbations and affects the dynamics in (at least) three ways, all related to non-local effects, i.e. to the interaction of a given fluid element with the environment. First, at the lowest perturbative order the shear is related to the tidal field generated by the surrounding material by a simple proportionality law. Second, it is related to a *dynamical* tidal induction: the modification of the environment forces the fluid element to modify its shape and density. In Newtonian gravity, this is an *action-at-a-distance* effect, which starts to manifest itself in second-order perturbation theory as an inverse-Laplacian contribution to the velocity

potential (e.g. Catelan et al. 1995). Third, and most important here, a non-vanishing shear field leads to the generation of a traceless and divergenceless metric perturbation which can be understood as gravitational radiation emitted by non-linear perturbations. This contribution to the metric perturbations is statistically small on cosmologically interesting scales, but it becomes relevant whenever anisotropic (with the only exception of exactly one-dimensional) collapse takes place. In the Lagrangian picture such an effect already arises at the post-Newtonian (PN) level. Note that the two latter effects are only detected if one allows for non-scalar perturbations in physical quantities. Contrary to a widespread belief, in fact, the choice of scalar perturbations in the initial conditions is not enough to prevent tensor modes to arise beyond the linear regime in a GR treatment. Truly tensor perturbations are dynamically generated by the gravitational instability of initially scalar perturbations, independently of the initial presence of gravitational waves. This point is very clearly displayed in the GR Lagrangian second-order perturbative approach. The pioneering work in this field is by Tomita (1967), who calculated the gravitational waves π^α_β emitted by non-linearly evolving scalar perturbations in an Einstein-de Sitter background, in the synchronous gauge. Matarrese, Pantano & Saez (1994a,b) obtained an equivalent result but with a different formalism in comoving and synchronous coordinates.

Recently a number of different approaches to relativistic effects in the non-linear dynamics of cosmological perturbations have been proposed. Matarrese, Pantano & Saez (1993) proposed an algorithm based on neglecting the magnetic part of the Weyl tensor in the dynamics, obtaining strictly local fluid-flow evolution equations, i.e. the so-called “silent universe”. This formalism, however, cannot be applied to cosmological structure formation *inside* the horizon, where the non-local tidal induction cannot be neglected, i.e. the magnetic Weyl tensor H^α_β is non-zero, with the exception of highly specific initial configurations (Matarrese et al. 1994a; Bertschinger & Jain 1994; Bruni, Matarrese & Pantano 1995a; the dynamical role of H^α_β was also discussed by Bertschinger & Hamilton 1994 and Kofman & Pogosyan 1995). Rather, it is probably related to the non-linear dynamics of an irrotational fluid *outside* the (local) horizon (Matarrese et al. 1994a,b). One possible application (Bruni, Matarrese & Pantano 1995b), is in fact connected to the *Cosmic No-hair Theorem*. Matarrese & Terranova (1995) followed the more “conservative” approach of expanding the Einstein and continuity equations in inverse powers of the speed of light, which then defines a Newtonian limit and, at the next order, post-Newtonian corrections. Their approach differs from previous ones, because of the gauge choice: we used synchronous and comoving coordinates, because of which this approach can be called a Lagrangian one. Various approaches have been proposed in the literature, which are somehow related. A PN approximation has been followed by Futamase (1991) to describe the dynamics of a clumpy universe. Tomita (1991) used non-comoving coordinates in a PN approach to cosmological perturbations. Shibata & Asada (1995) recently developed a PN approach to cosmological perturbations, also using non-comoving coordinates. Kasai (1995) analyzed the non-linear dynamics of dust in the synchronous and comoving gauge.

2. Method

We consider a pressureless fluid with vanishing vorticity. Using synchronous and comoving coordinates, the line-element reads

$$ds^2 = a^2(\tau) [-c^2 d\tau^2 + \gamma_{\alpha\beta}(\vec{q}, \tau) dq^\alpha dq^\beta] , \quad (2)$$

where we have factored out the scale-factor of the isotropic FRW solutions.

By subtracting the isotropic Hubble-flow, we introduce a *peculiar velocity-gradient tensor* $\vartheta^\alpha_\beta = \frac{1}{2} \gamma^{\alpha\gamma} \gamma_{\gamma\beta}'$, where primes denote differentiation with respect to τ .

Thanks to the introduction of this tensor we can write the Einstein's equations in a cosmologically convenient form. The energy constraint reads

$$\vartheta^2 - \vartheta^\mu_\nu \vartheta^\nu_\mu + 4 \frac{a'}{a} \vartheta + c^2 (\mathcal{R} - 6\kappa) = 16\pi G a^2 \varrho_b \delta , \quad (3)$$

where $\mathcal{R}^\alpha_\beta(\gamma)$ is the conformal Ricci curvature of the three-space with metric $\gamma_{\alpha\beta}$; for the background FRW solution $\gamma_{\alpha\beta}^{FRW} = (1 + \frac{\kappa}{4} q^2)^{-2} \delta_{\alpha\beta}$, one has $\mathcal{R}^\alpha_\beta(\gamma^{FRW}) = 2\kappa \delta^\alpha_\beta$. We also introduced the density contrast $\delta \equiv (\varrho - \varrho_b)/\varrho_b$.

The momentum constraint reads

$$\vartheta^\alpha_{\beta||\alpha} = \vartheta_{,\beta} . \quad (4)$$

The double vertical bars denote covariant derivatives in the three-space with metric $\gamma_{\alpha\beta}$.

Finally, after replacing the density from the energy constraint and subtracting the background contribution, the extrinsic curvature evolution equation becomes

$$\vartheta^\alpha_{\beta'} + 2 \frac{a'}{a} \vartheta^\alpha_\beta + \vartheta \vartheta^\alpha_\beta + \frac{1}{4} \left(\vartheta^\mu_\nu \vartheta^\nu_\mu - \vartheta^2 \right) \delta^\alpha_\beta + \frac{c^2}{4} \left[4\mathcal{R}^\alpha_\beta - (\mathcal{R} + 2\kappa) \delta^\alpha_\beta \right] = 0 . \quad (5)$$

The Raychaudhuri equation for the evolution of the *peculiar volume-expansion scalar* ϑ reads

$$\vartheta' + \frac{a'}{a} \vartheta + \vartheta^\mu_\nu \vartheta^\nu_\mu + 4\pi G a^2 \varrho_b \delta = 0 . \quad (6)$$

The main advantage of this formalism is that there is only one dimensionless (tensor) variable in the equations, namely the spatial metric tensor $\gamma_{\alpha\beta}$. The only remaining variable is the density contrast which can be written in the form

$$\delta(\vec{q}, \tau) = (1 + \delta_0(\vec{q})) [\gamma(\vec{q}, \tau) / \gamma_0(\vec{q})]^{-1/2} - 1 , \quad (7)$$

where $\gamma \equiv \det \gamma_{\alpha\beta}$.

3. Results and conclusions

The method is then based on a $1/c^2$ expansion of equations above which first of all leads to a new, purely Lagrangian, derivation of the Newtonian approximation

(Matarrese & Terranova 1995). One of the most important result in this respect is that we obtained a simple expression for the Lagrangian metric; exploiting the vanishing of the spatial curvature in the Newtonian limit we were able to write it in terms of the displacement vector $\vec{S}(\vec{q}, \tau) = \vec{x}(\vec{q}, \tau) - \vec{q}$, from the Lagrangian coordinate \vec{q} to the Eulerian one \vec{x} of each fluid element (e.g. Buchert 1995 and references therein), namely

$$ds^2 = a^2(\tau) \left[-c^2 d\tau^2 + \delta_{AB} \left(\delta^A_\alpha + \frac{\partial S^A(\vec{q}, \tau)}{\partial q^\alpha} \right) \left(\delta^B_\beta + \frac{\partial S^B(\vec{q}, \tau)}{\partial q^\beta} \right) \right]. \quad (8)$$

A straightforward application of this formula is related to the Zel'dovich approximation. The spatial metric is that of Euclidean space in time-dependent curvilinear coordinates, consistently with the intuitive notion of Lagrangian picture in the Newtonian limit. Read this way, the complicated equations of Newtonian gravity in the Lagrangian picture become much easier: one just has to deal with the spatial metric tensor and its derivatives. The displacement vector is then completely fixed by solving the Raychaudhuri equation together with the momentum constraint in the $c \rightarrow \infty$ limit.

Next, we can consider the post-Newtonian corrections to the metric and write equations for them. In particular, we can derive a simple and general equation for the gravitational-waves $\pi_{\alpha\beta}$ emitted by non-linear structures described through Newtonian gravity. The result can be expressed both in Lagrangian and Eulerian coordinates. In the latter case one has,

$$\nabla_x^2 \pi_{AB} = \Psi_{v,AB}^{(E)} + \delta_{AB} \nabla_x^2 \Psi_v^{(E)} + 2 \left(\bar{\vartheta} \bar{\vartheta}_{AB} - \bar{\vartheta}_{AC} \bar{\vartheta}^C_B \right), \quad (9)$$

with capital latin labels $A, B, \dots = 1, 2, 3$ indicating Eulerian coordinates and $\nabla_x^2 \Psi_v^{(E)} = -\frac{1}{2}(\bar{\vartheta}^2 - \bar{\vartheta}^A_B \bar{\vartheta}^B_A)$, which generally allows a simple derivation of π_{AB} , given the (gradients of the) velocity potential, $\bar{\vartheta}_{AB} = \partial^2 \Phi_v / \partial x^A \partial x^B$, by a convolution in Fourier space. These formulae would allow to calculate the amplitude of the gravitational-wave modes in terms of the velocity potential, which in turn can be deduced from observational data on radial peculiar velocities of galaxies.

In the standard case, where the cosmological perturbations form a homogeneous and isotropic random field, we can obtain a heuristic perturbative estimate of their amplitude in terms of the *rms* density contrast and of the ratio of the typical perturbation scale λ to the Hubble radius $r_H = cH^{-1}$. One simply has $\pi_{rms}/c^2 \sim \delta_{rms}^2 (\lambda/r_H)^2$. This effect gives rise to a stochastic background of gravitational waves which gets a non-negligible amplitude in the so-called *extremely-low-frequency* band (e.g. Thorne 1995), around $10^{-14} - 10^{-15}$ Hz. We can roughly estimate that the present-day closure density of this gravitational-wave background is

$$\Omega_{gw}(\lambda) \sim \delta_{rms}^4 \left(\frac{\lambda}{r_H} \right)^2. \quad (10)$$

In standard scenarios for the formation of structure in the universe, the typical density contrast on scales $1 - 10$ Mpc implies that Ω_{gw} is about $10^{-5} - 10^{-6}$. We might speculate that such a background would give rise to secondary CMB anisotropies on intermediate angular scales: a sort of *tensor Rees-Sciama effect*. This issue will be considered in more detail elsewhere.

The previous PN formula also applies to isolated structures, where the density contrast can be much higher than the *rms* value, and shear anisotropies play a fundamental role. A calculation of $\pi_{\alpha\beta}$ in the case of a homogeneous ellipsoid showed that the PN tensor modes become dominant, compared to the Newtonian contributions to the metric tensor, during the late stages of collapse, and possibly even in a shell-crossing singularity. It is important to stress that this effect generally contradicts the standard paradigm that the smallest scale for the applicability of the Newtonian approximation is set by the Schwarzschild radius of the object. Such a critical scale is indeed only relevant for nearly spherical collapse, whereas this effect becomes important if the collapsing structure strongly deviates from sphericity.

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